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# Boxed skew plane partition and integrable phase model

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## Abstract

We study the relation between the boxed skew plane partition and the integrable phase model. We introduce a generalization of a scalar product of the phase model and calculate it in two ways: the first one in terms of the skew Schur functions, and the other one by use of the commutation relations of operators. In both cases, a generalized scalar product is expressed as a determinant. We show that a special choice of the spectral parameters of a generalized scalar product gives the generating function of the boxed skew plane partition.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Interplay between statistical mechanics and combinatorics is one of the most interesting features in mathematical physics [1]. Recently, a connection between the integrable phase model and enumeration of plane partition was shown in [6]. The integrable phase model is a special limit of the  $q$ -boson model which is also related to the XXZ model [15–17], and it can be solved applying the quantum inverse scattering method (QISM) [9, 10].

Study of the generating function of plane partition [3–5] has attracted many researchers going back to MacMahon. For an illuminating story of plane partition involving alternating sign matrix theorems and the six-vertex model with domain wall boundary condition [11, 12], we refer to [2]. Recently, correlation functions for random (skew) plane partition were studied in a more general setting in terms of the free-fermion fields [7, 8]. There the generating function of unrestricted plane partition is presented as just a normalization constant, or a zero-point function.

In this paper, we generalize [6] and argue the connection between the integrable phase model and the boxed skew plane partition. The generating functions of the boxed skew plane

partition are given by a special parameter choice of the generalized scalar products of the phase model. We calculate the generalized scalar products with generic parameters in two ways. Namely, the one is expressed in terms of the Schur function, the eigenfunction for the phase model, and the other is, through the relations from QISM, expressed in terms of a determinant of matrix with elements of two-point functions and one-point functions.

The paper is organized as follows. In section 2, we introduce the phase model as a quantum integrable system. The integrability of this model is understood in the context of the QISM. Through graphical interpretations, the action of the monodromy operators on a Fock state turns out to be written in terms of the skew Schur functions. A scalar product, which is defined as the inner product of the  $N$ -particle states of the phase model, is generalized. The inhomogeneous phase model is also considered. In section 3, we briefly introduce the skew plane partition in a box and its generating function. In section 4, we see the connection between a generalized scalar product of the phase model and the skew plane partition. A sequence of operators in a generalized scalar product corresponds to the shape of the skew plane partition, and the number of lattice sites is equal to the height. The special choice of the spectral parameters naturally gives the generating function of the skew plane partition through graphical observations. A generalized scalar product is calculated in two ways in sections 5 and 6. The first one is by the help of the expression in the skew Schur functions. The other one is an inductive application of the commutation relations of the monodromy operators. In section 7, we show that the generating function of the boxed plane partition has a determinant expression, which is a by-product of the explicit expression of a generalized scalar product obtained in section 6. Section 8 is devoted to the conclusion.

## 2. The integrable phase model

In this section, we introduce the integrable phase model and review the results on this model concerning its relation to the skew Schur functions based on [6]. We also introduce and generalize the scalar product of the wavefunctions of the phase model. This is the main quantity which is viewed as the generating function of the plane partition in the subsequent sections.

### 2.1. Integrability of the phase model

We introduce the phase model as an integrable system. Consider a one-dimensional lattice of length  $M + 1$  and label each site by  $i = 0, \dots, M$ . Let  $\phi_i, \phi_i^\dagger$  denote the operators satisfying the following commutation relations:

$$[N_i, \phi_j] = -\phi_i \delta_{ij}, \quad [N_i, \phi_j^\dagger] = \phi_i^\dagger \delta_{ij}, \quad [\phi_i, \phi_j^\dagger] = \pi_i \delta_{ij} \quad (1)$$

for  $i, j = 0, \dots, M$  where  $N_i$  is the number operator and  $\pi_i$  is the vacuum projector. The  $L$ -operator for the phase model is defined as

$$L_i(u) = \begin{pmatrix} u^{-1} & \phi_i^\dagger \\ \phi_i & u \end{pmatrix} \quad (2)$$

with the spectral parameter  $u \in \mathbb{R}$ . The local Fock states at site  $j$  are created from the local vacuum state  $|0\rangle_j$  such as  $|n_j\rangle_j = (\phi_j^\dagger)^{n_j} |0\rangle_j$ ,  $N_j |n_j\rangle_j = n_j |n_j\rangle_j$ . The states satisfy

$$\begin{aligned} \phi_j |0\rangle_j &= 0, \\ \phi_j |n_j\rangle_j &= |n_j - 1\rangle_j, & \phi_j^\dagger |n_j\rangle_j &= |n_j + 1\rangle_j, \\ N_j |n_j\rangle_j &= n_j |n_j\rangle_j. \end{aligned} \quad (3)$$

The inner product of two Fock states is normalized,  ${}_i \langle m | n \rangle_j = \delta_{ij} \delta_{nm}$ .

Let us introduce the  $R$ -matrix (a  $4 \times 4$  matrix) of the form

$$\begin{aligned} R_{11}(u, v) &= R_{44}(u, v) = f(v, u), \\ R_{22}(u, v) &= R_{33}(u, v) = g(v, u), \\ R_{23}(u, v) &= 1, \end{aligned} \tag{4}$$

where

$$f(v, u) = \frac{u^2}{u^2 - v^2}, \quad g(v, u) = \frac{uv}{u^2 - v^2}, \tag{5}$$

and the other elements are zero. Then we can find that the  $L$ -operator satisfies the intertwining relation

$$R(u, v)(L_n(u) \otimes L_n(v)) = (L_n(v) \otimes L_n(u))R(u, v). \tag{6}$$

The monodromy matrix is defined as

$$T(u) = L_M L_{M-1} \cdots L_0 = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{7}$$

Hereafter, we refer to the elements of the monodromy matrix as the monodromy operators. The commutation relations of the monodromy operators can be obtained from

$$R(u, v)(T(u) \otimes T(v)) = (T(v) \otimes T(u))R(u, v). \tag{8}$$

Several explicit relations are listed in the following:

$$[A(u), A(v)] = [B(u), B(v)] = [C(u), C(v)] = [D(u), D(v)] = 0, \tag{9}$$

$$g(v, u)C(u)A(v) = f(v, u)C(v)A(u), \tag{10}$$

$$g(v, u)D(u)B(v) = f(v, u)D(v)B(u), \tag{11}$$

$$C(u)B(v) = g(u, v)\{A(u)D(v) - A(v)D(u)\} = g(u, v)\{D(u)A(v) - D(v)A(u)\}, \tag{12}$$

$$C(u)A(v) = f(v, u)A(v)C(u) + g(u, v)A(u)C(v), \tag{13}$$

$$D(u)B(v) = f(v, u)B(v)D(u) + g(u, v)B(u)D(v). \tag{14}$$

Note that these commutation relations can be reduced from those of the  $q$ -boson model.

### 2.2. Graphical representation of operators

The monodromy operators  $A, B, C$  and  $D$  are expressed as the linear combination of products of elements of  $L$ -operators. For later convenience, we introduce a graphical representation of the  $L$ -operator and the monodromy matrix in this subsection.

We assign a vertex with two vertical arrowed edges to each element of  $L$ -operators. Each arrow is directed upwards or downwards. The vertex is indexed by  $i$  for the  $i$ th site. The pairs of arrows stand for  $\phi_i, \phi_i^\dagger, u$  and  $u^{-1}$ , respectively (see figure 1).

This vertex representation is useful for a calculation of products of  $L$ -operators. We attach vertices vertically by the edge when the direction of the arrow between two neighbouring vertices is the same. This gives a one-dimensional vertical graph. Then, each element of a product of  $L$ -operators can be represented as summation over graphs of all arrow configurations with certain fixed boundaries. For instance, the  $(1, 1)$ -element of  $L_i(u)L_j(u)$  is  $u^{-2} + \phi_i^\dagger \phi_j$  and the corresponding graphs have both boundary arrows upwards.

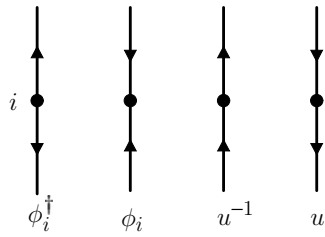


Figure 1. Vertex representation for the elements of the  $L$ -operator.

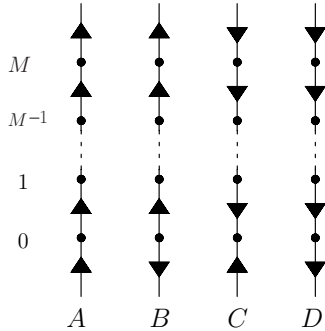


Figure 2. Basic arrow configurations of the operators  $A, B, C$  and  $D$ .

The monodromy operators  $A, B, C$  and  $D$  have also one-dimensional graph representations. From their definitions (7), each operator can be represented as a sum over graphs of all possible arrow configurations on a vertical lattice with  $M + 1$  vertices and fixed boundary arrows. Here the lattice sites are numbered by  $i = 0, \dots, M$  from the bottom. The operator  $B(u)$  (resp.  $C(u)$ ) has the top and bottom arrows pointing outwards (resp. inwards). The operator  $A(u)$  (resp.  $D(u)$ ) has the top and bottom arrows pointing upwards (resp. downwards).

The number of  $\phi^\dagger$  is bigger (resp. less) by one than that of  $\phi$  in every possible arrow configuration of  $B(u)$  (resp.  $C(u)$ ). In this sense, the operators  $B(u)$  and  $C(u)$  have the property of the creation and annihilation operator, respectively. On the other hand, the number of  $\phi^\dagger$  is exactly equal to that of  $\phi$  in every possible arrow configuration of  $A(u)$  and  $D(u)$ .

In later discussion, we call those shown in figure 2 the basic configurations. The other arrow configurations can be obtained by flipping some arrows upside-down.

2.3. Fock states and partitions

In this subsection, we see the actions of the monodromy operators  $A, B, C$  and  $D$  on a Fock state. We introduce a bijection from an arrow configuration considered in the previous subsection to a lattice path configuration. By using this bijection, we express each matrix element of the operators with respect to the Fock states in terms of the skew Schur functions.

Any state of the phase model can be expanded by a linear combination of states of the form

$$|\{n_j\}\rangle \equiv \prod_{j=0}^M |n_j\rangle_j, \quad N = \sum_{j=0}^M n_j, \quad n_j \geq 0, \tag{15}$$

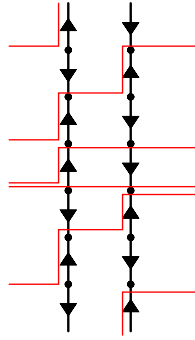


Figure 3. An example of a lattice path configuration corresponding to an arrow configuration.

where  $\{n_j\}$  stands for a configuration  $\{n_j\} = (n_0, \dots, n_M)$  with  $n_j \geq 0$ . We call this kind of state an  $N$ -particle basis. In particular, we denote the vacuum state by  $|0\rangle = \prod_{j=0}^M |0\rangle_j$ . As the action of the operator  $B(u)$  (resp.  $C(u)$ ) on a state increases (resp. decreases) the total number of particles by one, we call the state  $\prod_{j=1}^N B(u_j)|0\rangle$  the  $N$ -particle state and  $\langle 0| \prod_{j=1}^N C(u_j)$  its conjugate. Obviously, the  $N$ -particle state is a linear combination of  $N$ -particle bases. A matrix element  $\langle \{n'_j\} | B(u) | \{n_j\} \rangle$  vanishes unless  $|n'_j - n_j| \leq 1$  and  $N' = N + 1$ . Similarly, a matrix element  $\langle \{n'_j\} | C(u) | \{n_j\} \rangle$  vanishes unless  $|n'_j - n_j| \leq 1$  and  $N' = N - 1$ . Since the actions of  $A(w)$  and  $D(w)$  preserve the number of particles, matrix elements  $\langle \{n'_j\} | A(w) | \{n_j\} \rangle, \langle \{n'_j\} | D(w) | \{n_j\} \rangle$  vanish unless  $|n'_j - n_j| \leq 1$  and  $N' = N$ .

Now, we rephrase the arrow configuration of vertical graphs in the previous section as a non-intersecting up-right path configuration over the lattices. The action of the monodromy operator on a Fock state can be represented as a lattice path configuration on a vertical lattice.

Let us fix an  $N$ -particle basis  $\{n'_j\}$  and an arrow configuration of a vertical graph. Put  $n'_j$  horizontal paths entering the  $j$ th site from the left of the lattice. Put also another path entering the 0th site from the bottom if the bottom arrow is inward. The rule is that only one path can go along an upward arrow and no path along a downward arrow. If the  $j$ th site and the  $(j + 1)$ th site are connected by an upward arrow, one of the paths goes upwards from the  $j$ th site along the vertical lattice until it faces a downward arrow or another horizontal path, then it goes rightwards. Other paths getting into the  $j$ th site just pass rightwards horizontally. A path entering the  $M$ th site and going up along the top arrow, when the top arrow is upward, just goes vertically. Note that if there is no vertical path on a certain upward arrow, every  $|\{n_j\}\rangle$  is annihilated by one of  $\phi_j$ 's. As a result, we have a configuration  $\{n_j\}$  by counting  $n_j$  paths coming right out of the  $j$ th site. When the monodromy operators act successively on a Fock state, we can construct up-right paths by repeating the above procedure. In this way, a lattice path configuration corresponds one-to-one to an arrow configuration of vertical graphs. Figure 3 is an example of this correspondence.

We rewrite a Fock state in terms of a partition in order to obtain an explicit expression of the matrix elements of the monodromy operators. For a given  $N$ -particle basis  $|\{n_j\}\rangle$  we assume that  $n_{j_r} > 0$  ( $r = 1, \dots, R$ ) and  $M \geq j_1 > j_2 > \dots > j_R \geq 0$ . The identification is that  $\lambda = (\lambda_1, \dots, \lambda_N), M \geq \lambda_1 \geq \lambda_2 \dots \geq \lambda_N \geq 0$  where

$$\begin{aligned} \lambda_k &= j_1, & 1 \leq k \leq n_{j_1} \\ \lambda_k &= j_2, & n_{j_1} + 1 \leq k \leq n_{j_1} + n_{j_2} \end{aligned}$$

$$\dots$$

$$\lambda_k = j_R, \quad \sum_{r=1}^{R-1} n_{j_r} + 1 \leq k \leq \sum_{r=1}^R n_{j_r} = N. \quad (16)$$

We write the  $N$ -particle basis  $\{|n_j\rangle\}$  as  $|\lambda\rangle$ . The Fock space is spanned by the orthonormal basis  $|\lambda\rangle$  of partitions. In a lattice path configuration of the monodromy operator, the site  $\lambda_i$  is exactly the passing site of the  $i$ th path from above.

Then, the matrix elements of monodromy operators with respect to  $|\lambda\rangle$  bases have the following expression in terms of the skew Schur functions  $s_{\lambda/\mu}$ :

**Proposition 2.1.** *Matrix elements of the monodromy operators are given as follows ( $\lambda \subseteq M^N$ ).*

$$\begin{aligned} \langle \mu | A(u) | \lambda \rangle &= u^{-(M+1)} s_{\mu/\lambda}(u^2), & \text{for } \mu \subseteq M^N, \\ \langle \mu | B(u) | \lambda \rangle &= u^{-M} s_{\mu/\lambda}(u^2), & \text{for } \mu \subseteq M^{N+1}, \\ \langle \mu | C(u) | \lambda \rangle &= u^M s_{\lambda/\mu}(u^{-2}), & \text{for } \mu \subseteq M^{N-1}, \\ \langle \mu | D(u) | \lambda \rangle &= u^{M+1} s_{\lambda/\mu}(u^{-2}), & \text{for } \mu \subseteq M^N. \end{aligned} \quad (17)$$

**Proof.** We consider a non-vanishing  $\langle \mu | C(u) | \lambda \rangle$ . In the view of a lattice path configuration, the partitions  $\mu$  and  $\lambda$  are identified with the initial and final positions of the up-right paths, respectively. Fix  $\mu \subseteq M^{N-1}$  and consider the corresponding path configuration with respect to  $\lambda \subseteq M^N$ . We treat  $\lambda \subseteq M^N$  because another path comes in along the bottom arrow for the configuration of  $C(u)$ .

When, graphically, the arrow configuration for  $C(u)$  is the basic one, the path configuration is such that all  $N - 1$  paths from the left pass through without going vertically, and another path comes in from the bottom and goes out at site 0. Then,  $\lambda$  is equal to  $\mu$ . Note that  $\lambda_N = 0$  is irrelevant to the shape  $\lambda$ . We find in this case  $\langle \mu | C(u) | \lambda \rangle = u^M$ .

For the case where  $\lambda \neq \mu$ , the arrow configuration is the one where some downarrows except the boundary ones are flipped upside-down from the basic configuration. If one downarrow is flipped from the basic configuration, we have another factor  $u^{-2}$ . Let us consider a lattice path configuration where the  $i$ th path (from above) goes along  $l_i \geq 0$  uparrows. There are  $l = \sum l_i$  uparrows flipped upside-down from the basic configuration, and we find  $\langle \mu | C(u) | \lambda \rangle = u^{M-2l}$ . Now, the partitions  $\mu$  and  $\lambda$  satisfy the following properties. First,  $\lambda \supseteq \mu$  and  $\lambda_i = \mu_i + l_i$ . Second, the skew partition  $\lambda/\mu$  is a one-vertical strip since the condition that only one path should pass through every uparrow tells  $\lambda_{i+1} \leq \mu_i \leq \lambda_i, \forall i$ . From the definition of the skew Schur function,  $s_{\lambda/\mu}(u^{-2}) = u^{-2l}$ . Together with these observations, we finally obtain  $\langle \mu | C(u) | \lambda \rangle = u^M s_{\lambda/\mu}(u^{-2})$ .

Just similar considerations give matrix elements for the other operators.  $\square$

By inserting the complete set  $1 = \sum_v |v\rangle\langle v|$  between every successive operators and using  $\sum_v s_{\mu/v}(x) s_{v/\lambda}(y) = s_{\mu/\lambda}(x, y)$ , we furthermore have the following relations.

**Proposition 2.2.** *The matrix elements of products of the monodromy operators are as follows ( $\lambda, \lambda' \subseteq M^N, \mu \subseteq M^{N+J}$ ):*

$$\begin{aligned} \langle \mu | \prod_{j=1}^J B(u_j) | \lambda \rangle &= (u_1 \cdots u_J)^{-M} s_{\mu/\lambda}(u_1^2, \dots, u_J^2), \\ \langle \lambda | \prod_{j=1}^J C(u_j) | \mu \rangle &= (u_1 \cdots u_J)^M s_{\mu/\lambda}(u_1^{-2}, \dots, u_J^{-2}), \end{aligned}$$

$$\begin{aligned}
 \langle \lambda' | \prod_{j=1}^J A(u_j) | \lambda \rangle &= (u_1 \cdots u_J)^{-(M+1)} s_{\lambda'/\lambda}(u_1^2, \dots, u_J^2), \\
 \langle \lambda | \prod_{j=1}^J D(u_j) | \lambda' \rangle &= (u_1 \cdots u_J)^{M+1} s_{\lambda'/\lambda}(u_1^{-2}, \dots, u_J^{-2}).
 \end{aligned}
 \tag{18}$$

Other elements are zero.

These expressions allow us to write a generalized scalar product as a sum of products of the skew Schur functions (see section 5).

The Hermitian conjugates of the monodromy operators satisfy the following.

**Proposition 2.3.** *The Hermitian conjugates of the operators are related as*

$$\begin{aligned}
 A^\dagger(u) &= D(u^{-1}), & D^\dagger(u) &= A(u^{-1}), \\
 B^\dagger(u) &= C(u^{-1}), & C^\dagger(u) &= B(u^{-1}),
 \end{aligned}
 \tag{19}$$

i.e. for the monodromy matrix it holds that

$$T^\dagger(u) = T(u^{-1}).
 \tag{20}$$

**Proof.** From (17), we have  $\langle \mu | A(u) | \lambda \rangle = u^{-(M+1)} s_{\mu/\lambda}(u^2) = \langle \lambda | D(u^{-1}) | \mu \rangle$  and  $\langle \mu | B(u) | \lambda \rangle = u^{-M} s_{\mu/\lambda}(u^2) = \langle \lambda | C(u^{-1}) | \mu \rangle$ .  $\square$

#### 2.4. Generalized scalar product

We briefly review the scalar product of the phase model. The state  $\prod_{j=1}^N B(u_j) | 0 \rangle$  is the  $N$ -particle state. The inner product of  $N$ -particle states is called a scalar product of the phase model. A scalar product of the phase model was first calculated in [17] through the partition function of the six-vertex model with domain wall boundary condition [13, 14]. Another way to calculate it is to use the Schur function representation of the  $N$ -particle state. The explicit expression of the scalar product of the  $N$ -particle state is [6]

$$\begin{aligned}
 \langle 0 | \prod_{j=1}^N C(v_j) \prod_{j=1}^N B(u_j) | 0 \rangle &= \left( \prod_{i=1}^N \frac{u_i}{v_i} \right)^M \sum_{\lambda \subseteq M^N} s_\lambda(u_1^2, \dots, u_N^2) s_\lambda(v_1^{-2}, \dots, v_N^{-2}) \\
 &= \prod_{i < j} g(u_i, u_j) \prod_{i > j} g(v_i, v_j) \det[H(v_i, u_j)]_{1 \leq i, j \leq N},
 \end{aligned}
 \tag{21}$$

where

$$H(v_i, u_j) = \left\{ \left( \frac{u_j}{v_i} \right)^{M+N} - \left( \frac{v_i}{u_j} \right)^{M+N} \right\} \times \frac{1}{\frac{u_j}{v_i} - \frac{v_i}{u_j}}.
 \tag{22}$$

To prove (21), we just use the definition of the Schur function  $s_\lambda(x) = \prod_{i > j} (x_j - x_i)^{-1} \det x_i^{N-j+\lambda_j}$  when the length of  $\lambda$  is equal to or less than  $N$  and the Cauchy–Binet formula.

We define a generalized scalar product by the vacuum expectation value (VEV) of a sequence of the monodromy operators. It has a nonzero value only when the numbers of  $B$  and  $C$  are equal. In this paper, we consider only the generalized scalar product of a  $CB$ -type



word (see section 4.1):

$$\begin{aligned} & \langle 0 | \prod_{i=1}^{N_n} C(u_i^{C_n}) \prod_{i=1}^{M_n} A(u_i^{A_n}) \cdots \prod_{i=1}^{N_1} C(u_i^{C_1}) \prod_{i=1}^{M_1} A(u_i^{A_1}) \\ & \times \prod_{i=1}^{L_1} D(u_i^{D_1}) \prod_{i=1}^{K_1} B(u_i^{B_1}) \cdots \prod_{i=1}^{L_m} D(u_i^{D_m}) \prod_{i=1}^{K_m} B(u_i^{B_m}) | 0 \rangle. \end{aligned} \tag{23}$$

A generalized scalar product of this type can be calculated in two ways: by use of the skew Schur function representation of the phase model (see section 5), and by use of the commutation relations of the monodromy operators (see section 6). In both cases, a generalized scalar product can be expressed in a determinant form.

A scalar product of the phase model is related to the boxed plane partition by adjusting the parameters as shown in [6]. Below, we show that a generalized scalar product is related to the boxed skew plane partition. A sequence of operators appeared in (23) determines the base shape of the boxed skew plane partition. These subjects are considered in the successive sections.

### 2.5. Inhomogeneous phase model

We can generalize the phase model to that with inhomogeneous parameters; the spectral parameter  $u$  of  $L_j$  is replaced by  $uz_j$  in each lattice site  $j$ . Since  $f(vz_j, uz_j) = f(v, u)$  and  $g(vz_j, uz_j) = g(v, u)$ , the  $R$ -matrix is unchanged so that the relations for the monodromy matrix obtained from the QISM still remain the same. The only change that comes into our game is the functions

$$a(u; \{z_j\}) = u^{-(M+1)} \prod_{j=0}^M z_j^{-1}, \quad d(u; \{z_j\}) = u^{M+1} \prod_{j=0}^M z_j. \tag{24}$$

The calculation procedure for the generalized scalar product in this inhomogeneous case is much easier through the QISM than the explicit eigenfunctions in terms of the Schur function. A generalized scalar product for the inhomogeneous case has the same expression as that of the homogeneous case; the only difference is the functions (24).

## 3. Plane partition

### 3.1. Boxed plane partition

A plane partition is an array of non-negative numbers

$$\pi = (\pi_{ij}), \quad i, j = 1, 2, \dots,$$

that is non-increasing in both  $i$  and  $j$ . A plane partition is sometimes called a *3D partition*. Piling  $\pi_{ij}$  cubes over the  $(i, j)$  square in the two-dimensional square lattice gives a three-dimensional object. The volume of  $\pi$  is denoted by  $|\pi| = \sum \pi_{ij}$ . We call  $\pi$  a boxed plane partition (or a plane partition in a box) if  $1 \leq i \leq N$ ,  $1 \leq j \leq L$  and  $|\pi_{ij}| \leq M$  for some  $N, L$  and  $M$ . A boxed plane partition can be considered as stacks of cubes in a box with side lengths  $N, L, M$ . The generating function of the plane partition in an  $N \times L \times M$ -box is known to be

$$Z_q(N, L, M) = \sum_{\pi \in B(N, L, M)} q^{|\pi|} = \prod_{i=1}^N \prod_{j=1}^L \frac{1 - q^{i+j+M-1}}{1 - q^{i+j-1}}, \tag{25}$$

where  $B(N, L, M)$  stands for the  $N \times L \times M$ -box. If we take the limit  $N, L, M \rightarrow \infty$  with the assumption  $|q| < 1$ , the partition function  $Z_q$  converges to the celebrated MacMahon function,  $\prod_{n=1}^{\infty} (1 - q^n)^{-n}$ .

As explained in section 2.3, a scalar product of the phase model can be graphically interpreted in terms of lattice path configurations. A lattice path configuration is also viewed as a 2D-projection of a boxed (skew) plane partition through the lozenge tiling (see section 4.2); a plane partition in an  $N \times L \times M$ -box is also equivalent to a lozenge tiling of an  $(N, L, M)$ -semiregular hexagon. Hence there is a one-to-one correspondence between a boxed plane partition and a lattice path configuration.

The generating function of the boxed skew plane partitions can be naturally obtained by a special choice of parameters of the generalized scalar product. Up to some factor,  $\langle 0 | \prod_{j=1}^N C(v_j) \prod_{k=1}^N B(u_k) | 0 \rangle$  with  $u_j = q^{\frac{j-1}{2}}, v_j = q^{-\frac{j}{2}}, 1 \leq j \leq N$ , becomes the generating function of the plane partition in an  $N \times N \times M$ -box. Taking the corresponding generalized scalar product, an explicit determinant expression of (25) is given in section 7.

### 3.2. Boxed skew plane partition

We introduce a skew shape  $\lambda/\mu$  where  $\mu \subseteq \lambda$ . A skew plane partition  $\pi$  whose shape is  $\lambda/\mu$  is an array of non-negative integers  $(\pi_{ij})$  defined on the  $(i, j)$  squares contained in a skew shape  $\lambda/\mu$  that is non-increasing in both  $i$  and  $j$ . The volume of a skew plane partition is  $|\pi| = \sum \pi_{ij}$ . We call  $\pi$  a boxed skew plane partition when  $\lambda, \mu \subseteq N^L$  and  $|\pi_{ij}| \leq M$  for some  $N, L$  and  $M$ .

## 4. Skew plane partition and generalized scalar product

### 4.1. Skew shape from a sequence of operators

The scalar product of the  $N$ -particle states is relevant to the plane partition in an  $N \times N \times M$ -box. The base of the box is an  $N \times N$  square, corresponding to a sequence of the operators  $\prod_{j=1}^N C(v_j) \prod_{j=1}^N B(u_j)$ . More generally, a sequence of the operators for the generalized scalar product reads the shape of the boxed skew plane partition, as we will explain below.

Let us introduce a *word* consisting of four letters  $A, B, C$  and  $D$ . This is what ignores parameters of  $A(u), \dots, D(u)$  and vectors  $\langle 0 |$  and  $| 0 \rangle$  in the generalized scalar product. Let us define a *CB-type word* by a word in which the first  $N$  letters are  $A$  or  $C$ , and the next  $N'$  letters are  $B$  or  $D$  ( $N$  and  $N'$  are non-negative integers).  $CACADDDBB$  is an example of a *CB-type word*.

A generalized scalar product of a *CB-type word* corresponds to the plane partition with a skew shape  $\lambda/\mu$  where  $\lambda$  is a rectangle. We consider a *CB-type word* in which the numbers of  $A, B, C$  and  $D$  are  $N_A, N, N$  and  $N_D$ , respectively. Note that the numbers of  $B$  and  $C$  are equal to have a non-zero value for the generalized scalar product. The rectangle  $\lambda$  turns out to be  $(N + N_D) \times (N + N_A)$  (see figure 4). We put this rectangle on the two-dimensional lattice in a way that two diagonal corners are  $(0, 0)$  and  $(N + N_A, N + N_D)$ . Then, we draw a path as follows: (1) a path starts from the origin, (2) we read the word from left to right, and move one unit upwards (resp. rightwards) if the letter is either  $C$  or  $D$  (resp. either  $A$  or  $B$ ). The shape which is surrounded by two edges of the rectangle and the zig-zag path and contains the point  $(0, N + N_D)$  determines  $\mu$ . This is the one-to-one correspondence between a *CB-type word* and a boxed skew shape. Explicitly, for a generalized scalar product of a *CB-type word* (23)

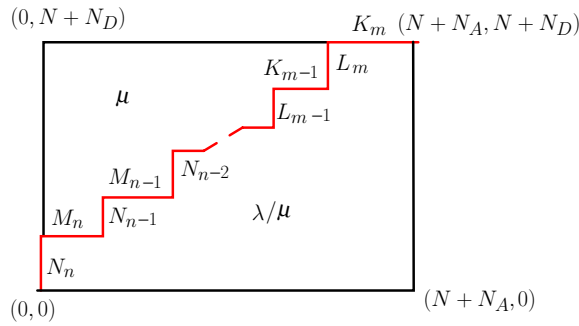


Figure 4. The skew shape determined by a sequence of the monodromy operators.

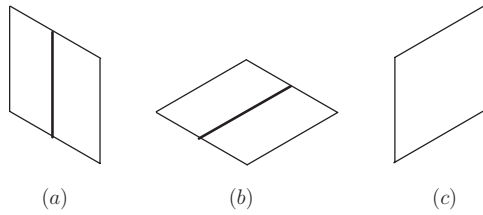


Figure 5. Three types of tiles.

the partition  $\mu$  is given by

$$\begin{aligned} \mu_j &= N + N_A - \sum_{p=1}^s K_{m+1-p}, & 1 + \sum_{p=1}^{s-1} L_{m+1-p} \leq j \leq \sum_{p=1}^s L_{m+1-p}, & \quad 1 \leq s \leq m, \\ \mu_j &= N + N_A - K - \sum_{q=1}^r M_q, & L + 1 + \sum_{q=1}^{r-1} N_q \leq j \leq L + \sum_{q=1}^r N_q, & \quad 1 \leq r \leq n, \end{aligned} \tag{26}$$

where we set  $N = \sum_{i=1}^n N_i$ ,  $N_A = \sum_{i=1}^n M_i$ ,  $N_D = \sum_{i=1}^m L_i$  and  $K = \sum_{i=1}^m K_i = N$ .

4.2. Lozenge tiling and skew plane partition

In this subsection, we consider a map from an arrow configuration on vertical graphs in section 2.3 to a lozenge tiling. A generalized scalar product with a special choice of the parameters can be naturally identified as the generating function of the skew plane partition.

Let us consider a lattice path configuration of a *CB*-type word. We arrange a lozenge tiling from this lattice path configuration. The domain of tiles is determined by the shape of the corresponding skew plane partition. Let one unit of a path going upwards correspond to a tile of type *a*, and let one unit of a path going rightwards correspond to a tile of type *b* in figure 5. The remained spaces are filled with tiles of type *c*. Figure 6 is an example of the correspondence among a lattice path configuration, an arrow configuration and a lozenge tiling. We also draw paths on the lozenge tiling to help understanding. These paths on a lozenge tiling are indeed non-intersecting by construction. We can see that only non-intersecting lattice path configurations are mapped to lozenge tilings.

A lozenge tiling gives a 2D-projection of a 3D piling of cubes in a box, and, therefore, a boxed skew plane partition.

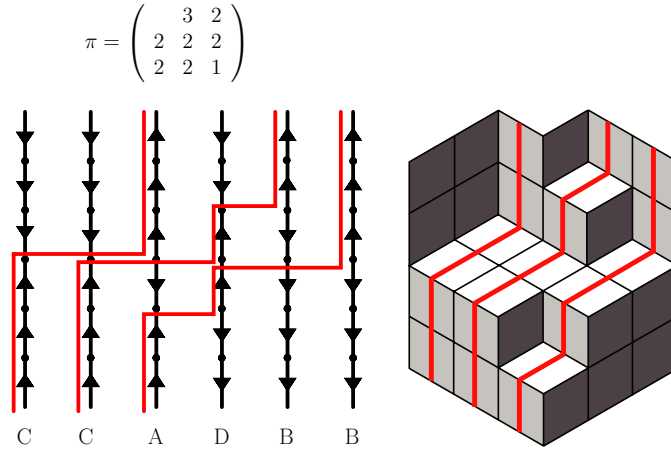


Figure 6. An example of identification of a skew plane partition with a lattice path configuration and a lozenge tiling.

In the arrow graph representation, we can show that the generalized scalar product of a CB-type word (23) is written as

$$\sum_c \prod_{n=1}^{N_c} (u_n^{C_c})^{M-2t_\uparrow^n} \prod_a \prod_{m=1}^{M_a} (u_m^{A_a})^{(M-1)-2t_\uparrow^m} \prod_b \prod_{l=1}^{L_b} (u_l^{B_b})^{2t_\downarrow^l - M} \prod_d \prod_{k=1}^{K_d} (u_k^{D_d})^{2t_\downarrow^k - (M-1)} \quad (27)$$

where the summation is taken over all arrow configurations, and  $t_\uparrow^i$  (resp.  $t_\downarrow^i$ ) counts the number of up (resp. down) arrows except the boundary arrows in the  $i$ th vertical graph. The powers of parameters are obtained by flipping some arrows upside-down from the basic configurations. Since the arrows at the boundaries are different, the powers of  $u^C$  and  $u^A$  (or  $u^D$  and  $u^B$ ) are different by one.

Suppose that  $\lambda$  is an  $(N + N_D) \times (N + N_A)$  rectangle,  $\mu$  is a partition (26) and  $\pi$  is a skew plane partition whose shape is  $\lambda/\mu$ . Using the correspondence of a skew plane partition and an arrow configuration, we obtain

$$|\pi| + M|\mu| + \frac{M}{2}(N_A - N_D)(N_A - N_D + 1) = \sum_{j=1}^{N+N_A} (-j)(-t_\uparrow^j) + \sum_{j=1}^{N+N_D} (j-1)t_\downarrow^j. \quad (28)$$

For  $1 \leq i \leq n$ , let  $s(A_i)$  be the set of  $M_i$  successive integers from  $\sum_{j=1}^{i-1} M_j + \sum_{j=1}^{i-1} N_j + 1$ ,  $s(A) = \cup_{i=1}^n s(A_i)$ , and let  $s(C_i)$  be the set of  $N_i$  successive integers from  $\sum_{j=1}^i M_j + \sum_{j=1}^{i-1} N_j + 1$ ,  $s(C) = \cup_{i=1}^n s(C_i)$ . Note that  $s(A) \cup s(C) = \{1, 2, \dots, N + N_A\}$ . This decomposition of integers corresponds to the product chain  $\prod^{N_n} C \prod^{M_n} A \cdots \prod^{N_1} C \prod^{M_1} A$ . Similarly, for  $1 \leq i \leq m$ , let  $s(D_i)$  be the set of  $L_i$  successive integers from  $\sum_{j=1}^{i-1} L_j + \sum_{j=1}^{i-1} K_j + 1$ ,  $s(D) = \cup_{i=1}^m s(D_i)$ , and let  $s(B_i)$  be the set of  $K_i$  successive integers from  $\sum_{j=1}^i L_j + \sum_{j=1}^{i-1} K_j + 1$ ,  $s(B) = \cup_{i=1}^m s(B_i)$ . Note the relations

$$\begin{aligned} \sum_{j \in s(C)} j + \sum_{j \in s(B)} (j-1) &= |\lambda/\mu|, \\ \sum_{j \in s(A)} j + \sum_{j \in s(D)} (j-1) &= |\mu| + \frac{1}{2}(N_A - N_D)(N_A - N_D + 1). \end{aligned} \quad (29)$$

Then from (28), we can show the following relation:

$$\begin{aligned} & \sum_{j \in s(C)} \binom{-j}{2} (M - 2t_{\uparrow}^j) + \sum_{j \in s(A)} \binom{-j}{2} (M - 1 - 2t_{\uparrow}^j) \\ & \quad + \sum_{j \in s(B)} \binom{j-1}{2} (2t_{\downarrow}^j - M) + \sum_{j \in s(D)} \binom{j-1}{2} (2t_{\downarrow}^j - M + 1) \\ & = |\pi| + \frac{2M+1}{2} |\mu| - \frac{M}{2} |\lambda| + \frac{M+1}{4} (N_A - N_D)(N_A - N_D + 1). \end{aligned} \tag{30}$$

We parameterize  $u^A, u^B, u^C$  and  $u^D$  as

$$\begin{aligned} \{u^{A_i}\} &= \{q^{-\frac{i}{2}} | j \in s(A_i)\}, & \text{for } 1 \leq i \leq n \\ \{u^{B_i}\} &= \{q^{\frac{i-1}{2}} | j \in s(B_i)\}, & \text{for } 1 \leq i \leq m \\ \{u^{C_i}\} &= \{q^{-\frac{i}{2}} | j \in s(C_i)\}, & \text{for } 1 \leq i \leq n \\ \{u^{D_i}\} &= \{q^{\frac{i-1}{2}} | j \in s(D_i)\}, & \text{for } 1 \leq i \leq m. \end{aligned} \tag{31}$$

We denote by  $Z_{GSP}(q^\bullet)$  the generalized scalar product (23) with the parameterization (31). Combining (27) and (30), we find that the generating function of a boxed skew plane partition is equal to

$$q^{-\alpha} Z_{GSP}(q^\bullet),$$

where

$$\alpha = \frac{2M+1}{2} |\mu| - \frac{M}{2} |\lambda| + \frac{M+1}{4} (N_A - N_D)(N_A - N_D + 1).$$

We give a remark about the upside-down property. The Hermitian conjugate of a generalized scalar product of a  $CB$ -type word yields the generating function of another kind of plane partition whose base shape is a hook Young diagram. From proposition 2.3, the Hermitian conjugate of the monodromy operators makes the arrow graphs upside-down and makes a lattice path configuration and a lozenge tiling rotate  $180^\circ$ . See figure 6 rotated  $180^\circ$ .

### 5. Generalized scalar product via skew Schur functions

#### 5.1. Cauchy–Binet formula

In the following, we present a formula for the generalized scalar product that is a determinant of the skew Schur functions. For this purpose, the basic is the Cauchy–Binet formula for calculating determinant.

Let two matrices  $A$  and  $B$  be of size  $m \times n$  and  $n \times m$ , respectively (without loss of generality,  $n \geq m$ ). Let  $S_{mn}$  denote the set of strictly increasing sequences of length  $m$  that can be chosen from  $\{1, 2, \dots, n\}$ . For any matrix  $X$  of size  $n \times m$  and any  $S = \{s_1, \dots, s_m\} \in S_{mn}$ , we denote by  $X(S|m)$  the  $m \times m$  matrix obtained from  $X$  by choosing all  $m$  columns of  $X$  and the  $m$  rows numbered by  $S$ , i.e.  $(X(S|m))_{ij} = X_{s_i, j}$ . Similarly, if  $X$  is  $m \times n$ , denote the  $m \times m$  matrix  $X_{i, s_j}$  ( $1 \leq i, j \leq m$ ) by  $X(m|S)$ . The Cauchy–Binet formula shows that the determinant of the product of two rectangular matrices  $A$  and  $B$  is

$$\det(AB) = \sum_{S \in S_{mn}} \det(B(S|m)) \det(A(m|S)). \tag{32}$$

5.2. Skew Schur functions

For any partition  $\lambda = (\lambda_1, \dots, \lambda_N)$ ,  $\lambda_i \geq 0$  and some non-negative integer  $L$ , we define  $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_{N+L})$  as

$$\tilde{\lambda}_i = \begin{cases} \tilde{\lambda}_i = i, & 1 \leq i \leq L \\ \tilde{\lambda}_i = \lambda_{N+L+1-i} + i, & L + 1 \leq i \leq N + L. \end{cases}$$

For any partitions  $\lambda = (\lambda_1, \dots, \lambda_N)$  and  $\mu \subseteq \lambda$ , the skew Schur function has a determinant expression of the form (*Jacobi–Trudi identity*)

$$\begin{aligned} s_{\lambda/\mu} &= \det (h_{\lambda_i - \mu_j - i + j})_{1 \leq i, j \leq N} \\ &= \det (h_{\tilde{\lambda}_i - \tilde{\mu}_j})_{1 \leq i, j \leq N+L}. \end{aligned} \tag{33}$$

where  $h_n, n \in \mathbb{Z}$ , is the complete symmetric function. Applying the Cauchy–Binet formula (32), we can show the following expressions.

**Proposition 5.1.** *For non-negative integers  $N, L$  and  $M$ , we have*

$$\sum_{\lambda \subseteq M^N} s_{\lambda/\mu}(x) s_{\lambda/\nu}(y) = \det \left[ \sum_{l=1}^{N+M+L} h_{l-\tilde{\mu}_i}(x) h_{l-\tilde{\nu}_j}(y) \right]_{1 \leq i, j \leq N+L}, \tag{34}$$

$$\sum_{\lambda \subseteq M^N} s_{\mu/\lambda}(x) s_{\nu/\lambda}(y) = \det \left[ \sum_{l=1}^{N+M+L} h_{-l+\tilde{\mu}_i}(x) h_{-l+\tilde{\nu}_j}(y) \right]_{1 \leq i, j \leq N+L}. \tag{35}$$

More generally, let  $\lambda^i = (\lambda_1^i, \dots, \lambda_{N_i}^i)$ ,  $0 \leq i \leq 2n$ , and let  $x^j = (x_1^j, \dots, x_{L_j}^j)$ ,  $1 \leq j \leq 2n$ , denote  $2n$  sets of variables; then we have a summation of products of the skew Schur functions over  $\sigma = \{\lambda^i | \lambda_1^i \leq M, 1 \leq i \leq 2n - 1\}$  as a determinant of complete symmetric functions.

$$\sum_{\sigma} \prod_{i=0}^{n-1} s_{\lambda^{2i+1}/\lambda^{2i}}(x^{2i+1}) s_{\lambda^{2i+1}/\lambda^{2i+2}}(x^{2i+2}) = \det [A_{\tilde{\lambda}_i^0 \tilde{\lambda}_j^{2n}}(x^1, \dots, x^{2n})]_{1 \leq i, j \leq N+L}, \tag{36}$$

where  $N = \max\{N_i | 1 \leq i \leq 2n\}$ ,  $L = \sum L_i$ , and

$$A_{l_0 l_{2n}}(x^1, \dots, x^{2n}) = \sum_{l_1, \dots, l_{2n-1}=1}^{N+L+M} \prod_{k=0}^{n-1} h_{l_{2k+1}-l_{2k}}(x^{2k+1}) h_{l_{2k+1}-l_{2k+2}}(x^{2k+2}). \tag{37}$$

Let us take the limit  $M \rightarrow \infty$ . The generating function of  $A_{ij}(x)$  becomes

$$\begin{aligned} \phi^A(x; z, w) &\equiv \sum_{i, j=1}^{\infty} z^i w^j A_{ij}(x^1, \dots, x^{2n}) \\ &= \left[ \frac{wz}{1-wz} \prod_{i=1}^n \prod_j (1 - x_j^i z^{-\sigma(i)})^{-1} \prod_{i=n+1}^{2n} \prod_j (1 - x_j^i w^{\sigma(i)})^{-1} \right]_+, \end{aligned} \tag{38}$$

where we take only positive powers and  $\sigma(i) = (-1)^{i+1}$ . Here, suppose that two functions  $(1 - \xi \zeta^{\pm 1})^{-1}$ ,  $\zeta = z, w$ , represent the formal power series of  $\zeta$ :  $(1 - \xi \zeta^{\pm 1})^{-1} = \sum_{j=0}^{\infty} \xi^j \zeta^{\pm j}$ .

If we also take partitions  $\lambda^0, \lambda^{2n} = \emptyset$ , we have

$$\begin{aligned} \sum_{\lambda^1, \dots, \lambda^{2n-1}} \prod_{i=0}^{n-1} s_{\lambda^{2i+1}/\lambda^{2i}}(x^{2i+1}) s_{\lambda^{2i+1}/\lambda^{2i+2}}(x^{2i+2}) &= \det [A_{ij}(x^1, \dots, x^{2n-1})]_{1 \leq i, j \leq N+L} \\ &= \prod_{i>j} \prod_{a>b} (1 - x_a^i x_b^j)^{-1}. \end{aligned} \tag{39}$$

5.3. Skew Schur function representation of generalized scalar products

Recall that matrix elements of the monodromy operators are expressed in terms of the skew Schur functions (17) and (18). From proposition 5.1, the generalized scalar product of the form (23) is written as

$$\begin{aligned} & \prod_{j=1}^n \frac{(\prod u^{C_j})^M}{(\prod u^{A_j})^{M+1}} \prod_{k=1}^m \frac{(\prod u^{D_k})^{M+1}}{(\prod u^{B_k})^M} \sum_{\lambda_i^1 \leq M, 1 \leq i \leq 2(n+m)-1} s_{\lambda^1}((u^{C_n})^{-2}) s_{\lambda^1/\lambda^2}((u^{A_n})^2) \cdots \\ & \quad \times s_{\lambda^{2n-1}/\lambda^{2n-2}}((u^{C_1})^{-2}) s_{\lambda^{2n-1}/\lambda^{2n}}((u^{A_1})^2) \\ & \quad \times s_{\lambda^{2n+1}/\lambda^{2n}}((u^{D_1})^{-2}) s_{\lambda^{2n+1}/\lambda^{2n+2}}((u^{B_1})^2) \cdots \\ & \quad \times s_{\lambda^{2(n+m)-1}/\lambda^{2(n+m)-2}}((u^{D_m})^{-2}) s_{\lambda^{2(n+m)-1}}((u^{B_m})^2) \\ & = \prod_{j=1}^n \frac{(\prod u^{C_j})^M}{(\prod u^{A_j})^{M+1}} \prod_{k=1}^m \frac{(\prod u^{D_k})^{M+1}}{(\prod u^{B_k})^M} \\ & \quad \times \det[A_{ij}((u^{C_n})^{-2}, (u^{A_n})^2, \dots, (u^{D_m})^{-2}, (u^{B_m})^2)]_{1 \leq i, j \leq N+L} \end{aligned}$$

where  $(u^{C_n})^{-2} = ((u_1^{C_n})^{-2}, \dots, (u_{N_n}^{C_n})^{-2})$  and so on. In particular, consider the plane partition in an  $N \times L \times \infty$ -box. This corresponds to  $\langle 0 | \prod_{k=L-N+1}^L C(q^{-k/2}) \prod_{k=1}^{L-N} A(q^{-k/2}) \prod_{k=1}^N B(q^{(k-1)/2}) | 0 \rangle$ . If we substitute the parameterization (31) into the above relation, we may reproduce (25) with  $M \rightarrow \infty$  by the help of (39), as expected.

6. Direct calculation of generalized scalar products

In the previous section, we have established a determinant formula for the generalized scalar product in terms of the skew Schur functions. In this section, we give a more explicit determinant formula. This is done by a direct application of the relations obtained from the QISM in section 2.1. The advantage of the formula here is a simplicity for evaluations. Actually in the next section, we demonstrate an evaluation of the determinant corresponding to the boxed plane partition in an  $N \times L \times M$ -box.

We remark about the notations. In this section, we often drop the indices of variables to avoid notational complexity if there is no confusion. For instance, we abbreviate  $\prod_{i=1}^N K(u_i^A)$  to  $\prod_{i=1}^N K(u^A)$  or just  $\prod K(u^A)$  and so on. Define the functions  $\Delta(u, v) = u^2 - v^2$ ,  $a(u) = u^{-(M+1)}$  and  $d(u) = u^{M+1}$ .

**Theorem 6.1.** *Generalized scalar products have the following expression in terms of a determinant:*

$$\begin{aligned} & \langle 0 | \prod_{i=1}^{N_n} C(u_i^{C_n}) \prod_{i=1}^{M_n} A(u_i^{A_n}) \cdots \prod_{i=1}^{N_1} C(u_i^{C_1}) \prod_{i=1}^{M_1} A(u_i^{A_1}) \\ & \quad \times \prod_{i=1}^{L_1} D(u_i^{D_1}) \prod_{i=1}^{K_1} B(u_i^{B_1}) \cdots \prod_{i=1}^{L_m} D(u_i^{D_m}) \prod_{i=1}^{K_m} B(u_i^{B_m}) | 0 \rangle \\ & = \prod_{i < j} g(u_i^{B \vee D}, u_j^{B \vee D}) \prod_{i > j} g(u_i^{A \vee C}, u_j^{A \vee C}) I_n(\{u^C\}; \{u^A\}) J_m(\{u^D\}; \{u^B\}) \\ & \quad \times (-1)^{LM} \det G(\{u^{A \vee C}\}; \{u^{B \vee D}\}), \end{aligned} \tag{40}$$

where  $G = (G_{i,j}^{a,b})$  is an  $(N + M + L) \times (N + M + L)$  matrix with

$$G_{i,j}^{a,b} = \begin{cases} H(u_i^{A \vee C}, u_j^{B \vee D}), & a = 0, \quad b = 0, \\ h_{b,j}(u_i^{A \vee C}), & 1 \leq i \leq N + M, \quad 1 \leq j \leq K + L (= N + L) \\ \tilde{h}_{a,i}(u_j^{B \vee D}), & a = 0, \quad 1 \leq b \leq n, \\ 0, & 1 \leq i \leq N + M, \quad 1 \leq j \leq M_b \\ \tilde{h}_{a,i}(u_j^{B \vee D}), & 1 \leq a \leq m, \quad b = 0, \\ 0, & 1 \leq i \leq L_a, \quad 1 \leq j \leq K + L \\ 0, & 1 \leq a \leq m, \quad 1 \leq b \leq n, \\ 0, & 1 \leq i \leq L_a, \quad 1 \leq j \leq M_b \end{cases} \quad (41)$$

and the functions are

$$h_{b,j}(u) = \prod_{\ell=1}^{b-1} \left[ \prod \Delta(u^{C_\ell}, u) \Delta(u^{A_\ell}, u) \right] a(u) u^{-2(j-1) - 2 \sum_{\ell=1}^{b-1} M_\ell - N} \quad (42)$$

$$\tilde{h}_{a,i}(u) = \prod_{\ell=1}^{a-1} \left[ \prod \Delta(u, u^{B_\ell}) \Delta(u, u^{D_\ell}) \right] d(u) u^{2(i-1) + \sum_{\ell=a}^m K_\ell - \sum_{\ell=1}^{a-1} K_\ell}$$

and

$$I_n(\{u^{C_1}\}, \dots, \{u^{C_n}\}; \{u^{A_1}\}, \dots, \{u^{A_n}\}) = \prod_{i=1}^n \left[ \left( \prod u^{C_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^n M_j} \left( \prod u^{A_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^n M_j - 1} \right] \quad (43)$$

$$J_m(\{u^{D_1}\}, \dots, \{u^{D_m}\}; \{u^{B_1}\}, \dots, \{u^{B_m}\}) = \prod_{i=1}^m \left[ \left( \prod u^{D_i} \right)^{-\sum_{j=1}^n L_j + 1} \left( \prod u^{B_i} \right)^{-\sum_{j=1}^n L_j} \right].$$

Here we set  $N = \sum N_i, M = \sum M_i, K = \sum K_i = N$  and  $L = \sum L_i$ . The sequence  $\{u_i^{A \vee C}\}_{i=1}^{N+M}$  is arranged as  $\{u_i^{C_n}\}_{i=1}^{N_n}$  followed by  $\{u_i^{A_n}\}_{i=1}^{M_n}, \dots, \{u_i^{C_1}\}_{i=1}^{N_1}, \{u_i^{A_1}\}_{i=1}^{M_1}$ . Similarly, the sequence  $\{u_i^{B \vee D}\}_{i=1}^{K+L}$  is arranged as  $\{u_i^{B_m}\}_{i=1}^{K_m}$  followed by  $\{u_i^{D_m}\}_{i=1}^{L_m}, \dots, \{u_i^{B_1}\}_{i=1}^{K_1}, \{u_i^{D_1}\}_{i=1}^{L_1}$ .

**Proof.** Firstly, since (9) holds, the generalized scalar product is symmetric in each of  $\{u^{A_i}\}, \{u^{B_i}\}, \{u^{C_i}\}$  and  $\{u^{D_i}\}$ , respectively. We can generalize relations (10) and (13) to

$$\prod_{i=1}^N C(u^C) \prod_{i=1}^M A(u^A) = \sum_{A', C'} \left( \prod \Delta(u^{C'}, u^{A'}) \right)^{-1} \times \left( \prod u^{C'} \right)^{2M-1} \left( \prod u^C \right) \prod_{i=1}^M A(u^{A'}) \prod_{i=1}^N C(u^{C'}). \quad (44)$$

Here the summation is taken over all decompositions  $\{u_i^{C'}\}_{i=1}^N \cup \{u_i^{A'}\}_{i=1}^M = \{u_i^C\}_{i=1}^N \cup \{u_i^A\}_{i=1}^M$ . Relation (44) is obtained as follows. first, the coefficient of  $\prod A(u^A) \prod C(u^C)$  is clearly  $\prod f(u^A, u^C)$ . To find the coefficient of  $\prod A(u^{A'}) \prod C(u^{C'})$ , exchange the variables of  $\prod C(u^C) \prod A(u^A)$  by relation (10) till  $\prod C(u^{C'}) \prod A(u^{A'})$ ; then move  $A$ 's through  $C$ 's and we have a coefficient

$$\prod_i \frac{g(u_i^{C A'}, u_i^{A C'})}{f(u_i^{C A'}, u_i^{A C'})} \prod f(u^{A'}, u^{C'}),$$



where each of  $u_i^{C A'} \in \{u^C\} \cap \{u^{A'}\}$  and  $u_i^{A C'} \in \{u^A\} \cap \{u^{C'}\}$  appears once in the product. By the definition (5) of  $f$  and  $g$ , we arrive at (44).

Fix the number of the operators  $C$  and  $B$  both  $N$ . For the case where there is no  $A$ 's and  $D$ 's, the formula is nothing but the scalar product (21). Assume the formula for  $n$  and  $m$ , that is,  $n$  products of  $A$ 's and  $m$  products of  $D$ 's are inserted. Then we calculate the case of  $n + 1$  insertion of products of  $A$ 's. Here we set  $N' = N_1 + \dots + N_n$ ,  $M' = M_1 + \dots + M_n$ . From (44) and the formula for  $n$  and  $m$ , we get

$$\begin{aligned}
 & \langle 0 | \prod_{i=1}^{N_{n+1}} C(u^{C_{n+1}}) \prod_{i=1}^{M_{n+1}} A(u^{A_{n+1}}) \dots \prod_{i=1}^{N_1} C(u^{C_1}) \prod_{i=1}^{M_1} A(u^{A_1}) \\
 & \quad \times \prod_{i=1}^{L_1} D(u^{D_1}) \prod_{i=1}^{K_1} B(u^{B_1}) \dots \prod_{i=1}^{L_m} D(u^{D_m}) \prod_{i=1}^{K_m} B(u^{B_m}) | 0 \rangle \\
 & = \sum_{A', C'} \left( \prod \Delta(u^{C'}, u^{A'}) \right)^{-1} \left( \prod u^{C'} \right)^{2M_{n+1}-1} \left( \prod u^{C_{n+1}} \right) \\
 & \quad \times \langle 0 | \prod_{i=1}^{M_{n+1}} A(u^{A'}) \prod_{i=1}^{N_{n+1}} C(u^{C'}) \prod_{i=1}^{N_n} C(u^{C_n}) \prod_{i=1}^{M_n} A(u^{A_n}) \dots \prod_{i=1}^{N_1} C(u^{C_1}) \prod_{i=1}^{M_1} A(u^{A_1}) \\
 & \quad \times \prod_{i=1}^{L_1} D(u^{D_1}) \prod_{i=1}^{K_1} B(u^{B_1}) \dots \prod_{i=1}^{L_m} D(u^{D_m}) \prod_{i=1}^{K_m} B(u^{B_m}) | 0 \rangle \\
 & = \prod_{i < j} g(u_i^{B \vee D}, u_j^{B \vee D}) \prod_{i > j} g(u_i^{A \vee C}, u_j^{A \vee C}) J_m(\{u^D\}; \{u^B\}) (-1)^{LM'} \\
 & \quad \times \sum_{A', C'} \det G(\{u^{A'c}\}; \{u^{B \vee D}\}) \prod a(u^{A'}) \times X, \tag{45}
 \end{aligned}$$

where  $\{u_i^{A'c}\}_{i=1}^{N+M'} = \{u_i^{A \vee C}\}_{i=1}^{N+M} \setminus \{u_i^{A'}\}_{i=1}^{M_{n+1}}$ , and the summation is taken over all decompositions  $\{u^{A'}\} \cup \{u^{C'}\} = \{u^{A_{n+1}}\} \cup \{u^{C_{n+1}}\}$ . Here we used  $\langle 0 | A(u) = \langle 0 | a(u)$ . We now calculate  $X$ :

$$\begin{aligned}
 X & = \left( \prod \Delta(u^{C'}, u^{A'}) \right)^{-1} \left( \prod u^{C'} \right)^{2M_{n+1}-1} \left( \prod u^{C_{n+1}} \right) \\
 & \quad \times \prod_{i > j} g(u_i^{A'c}, u_j^{A'c}) \left( \prod_{i > j} g(u_i^{A \vee C}, u_j^{A \vee C}) \right)^{-1} \\
 & \quad \times I_n(\{u^{C_1}\}, \dots, \{u^{C_n}\} \cup \{u^{C'}\}; \{u^{A_1}\}, \dots, \{u^{A_n}\}) \\
 & = \prod_{i < j} \Delta(u_i^{A \vee C}, u_j^{A \vee C}) \left( \Delta(u^{C'}, u^{A'}) \prod_{i < j} \Delta(u_i^{A'c}, u_j^{A'c}) \right)^{-1} \\
 & \quad \times I_n(\{u^{C_1}\}, \dots, \{u^{C_n}\} \cup \{u^{C'}\}; \{u^{A_1}\}, \dots, \{u^{A_n}\}) \\
 & \quad \times \left( \prod u^{A'c} \right)^{N'+N_{n+1}+M'-1} \left( \prod u^A \prod u^C \right)^{-(N'+N_{n+1}+M'+M_{n+1}-1)} \\
 & \quad \times \left( \prod u^{C'} \right)^{2M_{n+1}-1} \left( \prod u^{C_{n+1}} \right).
 \end{aligned}$$

We deal with the last expression in detail. First,

$$\begin{aligned}
 & \prod_{i < j} \Delta(u_i^{A \vee C}, u_j^{A \vee C}) \left( \prod_{i < j} \Delta(u^{C'}, u^{A'}) \prod_{i < j} \Delta(u_i^{A^c}, u_j^{A^c}) \right)^{-1} \\
 &= (-1)^\sigma \prod_{i < j} \Delta(u_i^{A'}, u_j^{A'}) \prod_{i < j} \Delta(u^{A^c}, u^{A'}) \left( \prod_{i < j} \Delta(u^{C'}, u^{A'}) \right)^{-1} \\
 &= (-1)^\sigma \prod_{i < j} \Delta(u_i^{A'}, u_j^{A'}) \prod_{i=1}^n \left[ \prod_{i=1}^n \Delta(u^{C_i}, u^{A'}) \prod_{i=1}^n \Delta(u^{A_i}, u^{A'}) \right] \\
 &= (-1)^\sigma \left( \prod u^{A'} \right)^{N'+N_{n+1}+2(M'+M_{n+1})-2} \\
 &\quad \times \det \left[ \prod_{i=1}^n \left[ \prod_{i=1}^n \Delta(u^{C_i}, u^{A'}) \prod_{i=1}^n \Delta(u^{A_i}, u^{A'}) \right] \right. \\
 &\quad \left. \times \left( u_k^{A'} \right)^{-(N'+N_{n+1}+2M'+2(\ell-1))} \right]_{1 \leq k, \ell \leq M_{n+1}}.
 \end{aligned}$$

In the last line, we used the Vandermonde determinant. Here,  $\sigma$  reorders  $\{u^{A \vee C}\}$  into the sequence  $\{u^{C'}\}$  followed by  $\{u^{C_n}\}, \{u^{A_n}\}, \dots, \{u^{A_1}\}, \{u^{A'}\}$ . Next,

$$\begin{aligned}
 & I_n(\{u^{C_1}\}, \dots, \{u^{C_n}\} \cup \{u^{C'}\}; \{u^{A_1}\}, \dots, \{u^{A_n}\}) \\
 &\quad \times \left( \prod u^{A^c} \right)^{N'+N_{n+1}+M'-1} \left( \prod u^A \prod u^C \right)^{-(N'+N_{n+1}+M'+M_{n+1}-1)} \left( \prod u^{C'} \right)^{2M_{n+1}-1} \\
 &\quad \times \left( \prod u^{C_{n+1}} \right) \left( \prod u^{A'} \right)^{N'+N_{n+1}+2(M'+M_{n+1})-2} \\
 &= \prod_{i=1}^n \left[ \left( \prod u^{C_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^n M_j} \left( \prod u^{A_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^n M_j - 1} \right] \left( \prod u^{C'} \right)^{M'} \\
 &\quad \times \left( \prod u^A \prod u^C \left( \prod u^{A'} \right)^{-1} \right)^{N'+N_{n+1}+M'-1} \left( \prod u^A \prod u^C \right)^{-(N'+N_{n+1}+M'+M_{n+1}-1)} \\
 &\quad \times \left( \prod u^{C'} \right)^{2M_{n+1}-1} \left( \prod u^{C_{n+1}} \right) \left( \prod u^{A'} \right)^{N'+N_{n+1}+2(M'+M_{n+1})-2} \\
 &= \prod_{i=1}^n \left[ \left( \prod u^{C_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^n M_j} \left( \prod u^{A_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^n M_j - 1} \right] \\
 &\quad \times \prod_{i=1}^n \left[ \prod u^{C_i} \prod u^{A_i} \right]^{-M_{n+1}} \left( \prod u^{C'} \prod u^{A'} \right)^{M'+M_{n+1}-1} \left( \prod u^{C_{n+1}} \right) \\
 &= \prod_{i=1}^n \left[ \left( \prod u^{C_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^{n+1} M_j} \left( \prod u^{A_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^{n+1} M_j - 1} \right] \\
 &\quad \times \left( \prod u^{C_{n+1}} \right)^{M'+M_{n+1}} \left( \prod u^{A_{n+1}} \right)^{M'+M_{n+1}-1} \\
 &= \prod_{i=1}^{n+1} \left[ \left( \prod u^{C_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^{n+1} M_j} \left( \prod u^{A_i} \right)^{\sum_{j=1}^i M_j - \sum_{j=i+1}^{n+1} M_j - 1} \right] \\
 &= I_{n+1}(\{u^{C_1}\}, \dots, \{u^{C_{n+1}}\}; \{u^{A_1}\}, \dots, \{u^{A_{n+1}}\}).
 \end{aligned}$$

Note that  $\prod u^{A'} \prod u^{C'} = \prod u^{A_{n+1}} \prod u^{C_{n+1}}$ . Going back to (45), substituting the above calculations, we get

$$\begin{aligned} & \prod_{i < j} g(u_i^B, u_j^B) \prod_{i > j} g(u_i^{A \vee C}, u_j^{A \vee C}) I_{n+1}(\{u^{C_1}, \dots, u^{C_{n+1}}\}; \{u^{A_1}, \dots, u^{A_{n+1}}\}) \\ & \quad \times J_m(\{u^D\}; \{u^B\}) (-1)^{LM'} \sum_{A', C'} (-1)^\sigma \det G(\{u^{A^c}\}; \{u^{B \vee D}\}) \prod a(u^{A'}) \\ & \quad \times \det \left[ \prod_{i=1}^n \left[ \prod \Delta(u^{C_i}, u_k^{A'}) \prod \Delta(u^{A_i}, u_k^{A'}) \right] \right. \\ & \quad \left. \times (u_k^{A'})^{-(N'+N_{n+1}+2M'+2(\ell-1))} \right]_{1 \leq k, \ell \leq M_{n+1}} \\ & = \prod_{i < j} g(u_i^B, u_j^B) \prod_{i > j} g(u_i^{A \vee C}, u_j^{A \vee C}) I_{n+1}(\{u^C\}; \{u^A\}) J_m(\{u^D\}; \{u^B\}) (-1)^{LM'} \\ & \quad \sum_{A', C'} (-1)^{\sigma'+LM_{n+1}} \det G(\{u^{A^c}\}; \{u^{B \vee D}\}) \det[h_{n+1, \ell}(u_k^{A'})]_{\substack{1 \leq k \leq M_{n+1} \\ N+M'+1 \leq \ell \leq N+M'+M_{n+1}}} \end{aligned}$$

where  $\sigma'$  sends the indices of row with  $\{u^{A'}\}$  for matrix  $G(\{u^{A \vee C}\}; \{u^{B \vee D}\})$  to the last. There appears another sign  $(-1)^{LM_{n+1}}$  because  $L$  lines of the matrix  $G$  with  $1 \leq a \leq m$  are passed through by  $M_{n+1}$  lines. We can extend the summation of  $A', C'$  to all  $\{u^{A'}\} \subset \{u^{A \vee C}\}$  since  $\det[h_{n+1, \ell}(u_k^{A'})] = 0$  unless  $\{u^{A'}\} \subset \{u^{A_{n+1}}\} \cup \{u^{C_{n+1}}\}$ . As a result, the sum becomes

$$(-1)^{LM_{n+1}} \det G(\{u^{A \vee C}\}; \{u^{B \vee D}\}).$$

The whole output is exactly the  $(n + 1)$ th case of the formula. Computation for the  $(m + 1)$ th case is done in the same way by developing relations (11) and (14). Then by induction the proof is completed.  $\square$

### 7. Generating function of boxed plane partition

As an application of the formula in the previous section, we evaluate the determinant of the generalized scalar product corresponding to the generating function of the boxed plane partition in an  $N \times L \times M$ -box. We can see the appearance of the function (25) in the following proposition.

**Proposition 7.1.** *The following product expression holds for the generalized scalar product:*

$$\begin{aligned} & \langle 0 | \prod_{k=L-N+1}^L C(q^{-k/2}) \prod_{k=1}^{L-N} A(q^{-k/2}) \prod_{k=1}^N B(q^{(k-1)/2}) | 0 \rangle \\ & = q^{-\frac{1}{2}LMN + \frac{1}{4}(L-N)(M+1)(L-N+1)} \prod_{i=1}^L \prod_{j=1}^N \frac{1 - q^{i+j+M-1}}{1 - q^{i+j-1}}. \end{aligned} \tag{46}$$

**Proof.** For an evaluation of the relevant determinant in (40), we use the following determinant formula extending the one in [11]:

$$\begin{aligned} & \det \left[ \left( \frac{s^{(i+j-1)/2} - s^{-(i+j-1)/2}}{q^{(i+j-1)/2} - q^{-(i+j-1)/2}} \right)_{\substack{1 \leq i \leq L \\ 1 \leq j \leq N}} \left( (sq^{2j-1})^{i/2} \right)_{\substack{1 \leq i \leq L \\ 1 \leq j \leq L-N}} \right] \\ & = (-1)^{\frac{1}{2}N(N-1)} q^{\frac{1}{4}(L-N)(L^2+N^2-LN+L-N)} s^{\frac{1}{4}(L-N)(L-N+1)} \end{aligned}$$

$$\begin{aligned}
 & \times (q^{1/2} - q^{-1/2})^{\frac{1}{2}L(L-1) + \frac{1}{2}N(N-1)} \\
 & \times \prod_{k=0}^{N-1} [\sigma - k]^{N-k} \prod_{k=1}^{L-N} [\sigma + k]^N \prod_{k=L-N+1}^{L-1} [\sigma + k]^{L-k} \\
 & \times \prod_{1 \leq j < i \leq N} [i - j]^2 \prod_{N+1 \leq j < i \leq L} [i - j] \prod_{\substack{L-N+1 \leq i \leq L \\ 1 \leq j \leq N}} [i + j - 1]^{-1} \tag{47}
 \end{aligned}$$

where we put  $s = q^\sigma = q^{M+N}$  and  $[k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}$ . The remaining computation for the proof is a straightforward exercise.

Equation (47) is briefly proved as follows. The determinant of the matrix multiplied by  $s^{i/2}$  for rows with  $1 \leq i \leq L$  and  $s^{(j-1)/2}$  for columns with  $1 \leq j \leq N$  is a polynomial in  $s$  of degree  $LN + (L - N)(L - N + 1)/2$ . If we put  $s = q^k, 0 \leq k \leq N - 1$ , the matrix is spanned by linearly independent vectors  $\mathbf{x}_j = (q^{(2j+1-k)i/2}), 1 \leq j \leq k$ , and  $\mathbf{y}_j = (q^{(2j+1+k)i/2}), 1 \leq j \leq L - N$ , and its rank is  $k + L - N$ , so we have the divisor  $(s - q^k)^{N-k}$ . If we put  $s = q^{-k}, 1 \leq k \leq L - 1$ , the matrix is spanned by linearly independent vectors  $\mathbf{x}_j = (q^{(2j+1-k)i/2}), 1 \leq j \leq \max\{k, L - N\}$ , and its rank is  $\max\{k, L - N\}$ , so we have the divisor  $(s - q^{-k})^N$  for  $1 \leq k \leq L - N$  and  $(s - q^{-k})^{L-k}$  for  $L - N + 1 \leq k \leq L - 1$ . We only need the overall factor. It is obtained by examining the coefficient of the lowest degree term of  $s$ . It turns out to be the product of a Cauchy-type determinant and a Vandermonde-type determinant, so it can be easily calculated.  $\square$

### 8. Conclusion

In this paper, we have introduced and calculated a generalized scalar product of the integrable phase model. We have shown in two ways that a generalized scalar product with generic spectral parameters is expressed as a determinant. First, we have shown that the matrix elements of the monodromy operators are expressed in terms of the skew Schur functions through graphical interpretations, vertical lattices with arrows and lattice paths on them. A generalized scalar product is expressed as the sum of products of the skew Schur functions, where the sum is taken over all the partitions satisfying a restriction  $\lambda_1 \leq M$ . The sum is reexpressed as a determinant, applying the Cauchy–Binet formula. Second, the commutation relations for the monodromy operators of the QISM allow us to express a generalized scalar product as a determinant of matrix with elements of two-point functions and one-point functions. This expression is also valid for the inhomogeneous phase model.

We have found that a generalized scalar product is related to the generating function of the skew plane partition. More precisely, a generalized scalar product with the special choice of the spectral parameters is up to some factor equal to the generating function of the corresponding skew plane partition. A sequence of the monodromy operators determines the skew shape.

The skew plane partition in the semi-infinite box was considered in terms of the free fermion representation in [7, 8]. The generating function of the plane partition can be written as the expectation value of products of successive vertex operators. The phase model representation here of the skew plane partition has some advantages when restricted in a box. In the fermion formalism this restriction requires us to insert projection operators between vertex operators and makes it difficult to calculate the generating function explicitly. Correlation functions are easily treated in the fermion formalism. To find an explicit expression of general correlation functions in the phase model formalism is remained as a future problem.

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